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Diffusion Effects on the Breakdown of a Linear Amplifier Model Driven by the Square of a Gaussian Field

A. Asselah,[1] P. Dai Pra,[2] J. L. Lebowitz,[3] and Ph. Mounaix[4]

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We investigate solutions to the equation $\partial_t \mathscr{E} - \mathscr{D}\Delta\mathscr{E} = \lambda S^2 \mathscr{E}$, where $S(x, t)$ is a Gaussian stochastic field with covariance $C(x-x', t, t')$, and $x \in \mathbb{R}^d$. It is shown that the coupling $\lambda_{cN}(t)$ at which the N-th moment $\langle \mathscr{E}^N(x, t) \rangle$ diverges at time t, is always less or equal for $\mathscr{D} > 0$ than for $\mathscr{D} = 0$. Equality holds under some reasonable assumptions on C and, in this case, $\lambda_{cN}(t) = N\lambda_c(t)$ where $\lambda_c(t)$ is the value of λ at which $\langle \exp[\lambda \int_0^t S^2(0, s) ds] \rangle$ diverges. The $\mathscr{D} = 0$ case is solved for a class of S. The dependence of $\lambda_{cN}(t)$ on d is analyzed. Similar behavior is conjectured when diffusion is replaced by diffraction, $\mathscr{D} \to i\mathscr{D}$, the case of interest for backscattering instabilities in laser-plasma interaction.

KEY WORDS: Diffusion; Gaussian field; backscattering.

I. INTRODUCTION

We investigate the development of a linear amplification in a system driven by the square of a Gaussian noise. This problem arose and continues to be of interest in modeling the backscattering of an incoherent high intensity laser light by a plasma. There is a large litterature on this topic, and we refer the interested reader to ref. 1 for background. Our starting point here

[1] LATP, UMR 6632 du CNRS, Centre de Mathématique et Informatique, Université de Provence, 39 rue F. Joliot-Curie, 13453 Marseille Cedex 13, France.

[2] Dipartimento di Matematica, Politecnico di Milano Piazza Leonardo da Vinci 32, I-20133 Milano, Italy.

[3] Departments of Mathematics and Physics, Rutgers, The State University of New Jersey, Piscataway, New Jersey 08854-8019.

[4] Centre de Physique Théorique, UMR 7644 du CNRS, Ecole Polytechnique, 91128 Palaiseau Cedex, France.

is the work by Rose and Dubois[2] who investigated the following equation for the complex amplitude $\mathscr{E}(x, z)$ of the scattered electric field

$$\begin{cases} \partial_z \mathscr{E}(x, z) - i\mathscr{D}\Delta\mathscr{E}(x, z) = \lambda\, |S(x, z)|^2\, \mathscr{E}(x, z), \\ z \in [0, L],\ x \in \Lambda \subset \mathbb{R}^2, \text{ and } \mathscr{E}(x, 0) = \mathscr{E}_0(x). \end{cases} \tag{1}$$

In Eq. (1), z and x correspond to the axial and transverse directions in a plasma of length L and cross-sectional domain Λ. The input at $z = 0$, $\mathscr{E}_0(x)$, is a given function of x and Λ will be generally taken to be a torus (e.g. in numerical solutions of Eq. (1) using spectral methods). The coupling constant $\lambda > 0$ is proportional to the average laser intensity and $\mathscr{D}$ is a constant parameter introduced for convenience. The complex amplitude of the laser electric field $S(x, z)$ is a homogeneous Gaussian stochastic field defined by

$$\langle S(x, z)\rangle = \langle S(x, z) S(x', z')\rangle = 0,$$

$$\langle S(x, z)\, S(x', z')^* \rangle = C(x - x', z - z'),$$

where the correlation function $C(x, z)$ is the solution to

$$\begin{cases} \partial_z C(x, z) + \dfrac{i}{2}\, \Delta C(x, z) = 0, \\ z \in [0, L],\ x \in \Lambda, \text{ and } C(x, 0) = \mathscr{C}(x), \end{cases} \tag{2}$$

with $\mathscr{C}(x)$ a given function of x,[3] normalized so that $\mathscr{C}(0) \equiv \langle |S(x, z)|^2 \rangle = 1$.

Using heuristic arguments and numerical simulations, Rose and DuBois found that the expected value of the energy density of the scattered field $\langle |\mathscr{E}(x, L)|^2 \rangle$ diverged for every $L > 0$ as λ increased to some critical value $\lambda_c(L)$. The average $\langle |\mathscr{E}|^2 \rangle$ is over the realizations of the Gaussian field S. This divergence indicates a breakdown in the assumptions made in deriving Eq. (1), which neglects both nonlinear saturation and transient time evolution.[2,4] Physically, it can be interpreted as indicating a change in the nature of the amplification caused by the plasma.

To see the origin of this divergence in its simplest form, consider the case where $\mathscr{D}$ is set equal to zero in Eq. (1), and neglect all dependence of S on x and z. We are then led to the equation

$$\frac{d\mathscr{E}(z)}{dz} = \lambda S^2 \mathscr{E}(z), \tag{3}$$

which yields

$$\mathscr{E}(z) = \mathscr{E}(0)\, e^{\lambda S^2 z}, \qquad z > 0.$$

Here $S^2 = S_1^2 + S_2^2$ and S_1, S_2 are two independent real Gaussian random variables with zero mean and unit variance. It is easily seen that the probability distribution of $\mathscr{E}(z)$, setting $\mathscr{E}(0) = 1$, has the density

$$W(\mathscr{E}, z) = (2\lambda z)^{-1}\, \mathscr{E}^{-[1+(2\lambda z)^{-1}]} \qquad \text{for} \quad \mathscr{E} \geqslant 1, \quad z > 0. \tag{4}$$

If we now take moments of $\mathscr{E}$ at some value L of z, we find that $\langle \mathscr{E}^N(L) \rangle$ will diverge whenever $2N\lambda L \geqslant 1$. At the critical coupling $\lambda_{cN}(L) = (2NL)^{-1}$, there is a qualitative transition of the amplification of $\langle \mathscr{E}^N(L) \rangle$ from a regime where it is dominated by the bulk of the order-one-fluctuations of S to a regime where it is dominated by the large fluctuations of S. This toy model can be thought of as an idealization in which the size of the plasma is very small compared to the correlation length of the laser field. This is certainly not a reasonable physical approximation and we shall later consider situations in which S in Eq. (3) is z-dependent with a covariance $C(z, z')$. The equation is then still solvable more or less explicitly, depending on the form of C, at least as far as the dependence of the divergence of the moments of $\mathscr{E}$ on λ and L is concerned. The main difference from Eq. (4) is that for small enough values of λ, the first few moments need not diverge for any L.

In this paper, we extend these results to the x-dependent case where $i\mathscr{D}$ in Eq. (1) is replaced by $\mathscr{D}$, i.e. we consider a diffusive process in x rather than a diffractive one. Somewhat surprisingly the diffusion does not suppress the onset of divergences in moments of the field. This suggests a similar behavior for the diffractive case—in accord with the numerical results of ref. 2—but we are unable to prove this at the present time.

Before going on to the formulation and presentation of results for the diffusive case, we make some remarks about the relation between expectations over different realizations of the Gaussian driving term $|S|^2$ and the outcome of a given experiment. Accepting the idealizations inherent in assuming Gaussian statistics and neglect of nonlinear terms, the physically relevant question relating to the solution of the stochastic PDE (1) appears to be the following: What is the probability that for given Λ and L there will be small regions in Λ through which a significant fraction of the total incoming power is backscattered, (here "total" means through the whole domain Λ). Put more physically, imagine Λ to be divided up into $M \gg 1$ cells of equal area $|\Lambda|/M$ and let $R \gg 1/M$ be a specified number. We want to compute the probability P that in at least one of the cells the integral of $|\mathscr{E}|^2$ over that cell exceeds $R\,|\Lambda|$. In the case where $\mathscr{D}$ is set equal to zero, this can be answered by taking for the cell size the transverse correlation length of $|S|^2$ and assuming the field inside each cell to be transversally constant and evolving along z under Eq. (3) with a z-dependent S.

One finds that P greatly increases as λ passes its critical value for the divergence of the second moment, from $P \ll 1$ for $\lambda < \lambda_{c2}(L)$ to $P \simeq 1$ for $\lambda > \lambda_{c2}(L)$. We expect that this probability will behave similarly in real systems.

The outline of the rest of this paper is as follows. In Section 2 we introduce our diffusion-amplification model. In Section 3 we prove that the value of the critical coupling obtained without the diffusion term cannot be less than the one obtained with the diffusion term. In Section 4 we prove that for a large class of Gaussian fields S the values of the critical coupling obtained with or without the diffusion term are the same. Section 5 is devoted to the explicit solution of the diffusion-free problem in the particular case where the on-axis field $S(0, z)$ is a linear functional of a Gauss–Markov process. Finally, in Section 6 we study the dependence of the critical coupling on the space dimensionality in the case of a factorable correlation function C.

II. MODEL AND DEFINITIONS

As explained in the introduction, we consider a modified version of the linear convective amplifier model obtained by replacing $i\mathscr{D}$ by $\mathscr{D}$ on the left-hand side of Eq. (1). Taking $\mathscr{D} = 1/2$ without loss of generality, one is thus led to the problem

$$\begin{cases} \partial_t \mathscr{E}(x,t) - \frac{1}{2}\Delta \mathscr{E}(x,t) = \lambda S(x,t)^2\, \mathscr{E}(x,t), \\ t \in [0,T],\ x \in \mathbb{R}^d, \text{ and } \mathscr{E}(x,0) = \mathscr{E}_0(x), \end{cases} \tag{5}$$

where, following the usual notation used in diffusion problems, the time variable t (resp. T) plays the role of the axial space variable z (resp. L). In Eq. (5), we restrict ourselves to the cases where $S(x, t)$ is a real homogeneous Gaussian field defined by

$$\langle S(x,t)\rangle = 0,$$
$$\langle S(x,t)\, S(x',t')\rangle = C(x-x', t, t'),$$

with the normalization $C(0,0,0) \equiv \langle S(x,0)^2\rangle = 1$, and we will take $\mathscr{E}_0(x) \equiv 1$ as an initial condition. Note that $S(x, t)$ is not assumed to be stationary in t, and that the rest of our analysis is essentially unaffected if we replace $\mathbb{R}^d$ by a d-dimensional torus.

The critical coupling $\lambda_{cN}(T)$ and its diffusion-free counterpart $\bar{\lambda}_{cN}(T)$ are defined by

$$\lambda_{cN}(T) = \inf\{\lambda > 0 : \langle \mathscr{E}(0,T)^N\rangle = +\infty\}, \tag{6a}$$

$$\bar{\lambda}_{cN}(T) = \inf\{\lambda > 0 : \langle e^{N\lambda \int_0^T S(0,t)^2\, dt}\rangle = +\infty\}, \tag{6b}$$

where $\langle . \rangle$ denotes the average over the realizations of S. For a given $T > 0$, Eqs. (6) gives the value of λ at which $\langle \mathscr{E}(x, T)^N \rangle$ blows up with and without diffusion respectively.

Finally, in order not to make the calculations too cumbersome, we will use in the following the compact notation

$$\mathbf{t} \equiv (n, t),$$

$$\int d\mathbf{t} \equiv \sum_{n=1}^{N} \int_0^T dt,$$

$$S(\mathbf{t}) \equiv S(x_n(t), t),$$

$$C(\mathbf{s}, \mathbf{t}) \equiv \langle S(\mathbf{s})\, S(\mathbf{t}) \rangle = C(x_n(s) - x_m(t), s, t),$$

$$C_0(\mathbf{s}, \mathbf{t}) \equiv C(0, s, t),$$

$$(\varphi, \psi) = \int \varphi(\mathbf{t})\, \psi(\mathbf{t})\, d\mathbf{t},$$

with $s, t \in [0, T]$, $n, m \in \mathbb{N}$ $(1 \leqslant n, m \leqslant N)$, and where the $x_n(\cdot)$ are given continuous paths on $\mathbb{R}^d$. The covariance operators $\hat{T}_C$ and $\hat{t}_{C_0}$, respectively acting on $\varphi(\mathbf{t}) \in L^2(d\mathbf{t})$ and $\varphi(t) \in L^2(dt)$, are defined by

$$(\hat{T}_C \varphi)(\mathbf{s}) = \int C(\mathbf{s}, \mathbf{t})\, \varphi(\mathbf{t})\, d\mathbf{t},$$

$$(\hat{t}_{C_0} \varphi)(s) = \int_0^T C(0, s, t)\, \varphi(t)\, dt.$$

III. COMPARISON OF $\lambda_{cN}(T)$ AND $\bar{\lambda}_{cN}(T)$

In this section we prove that $\lambda_{cN}(T) \leqslant \bar{\lambda}_{cN}(T)$. We begin with two technical lemmas that will be useful in the following.

Lemma 1. Suppose the covariance function $C(x, t, t')$ is continuous. Let $\mu_1^{x(\mathbf{t})} \geqslant \mu_2^{x(\mathbf{t})} \geqslant \cdots \geqslant 0$ be the eigenvalues of the covariance operator $\hat{T}_C$. Here, the superscript $x(\mathbf{t})$ denotes the N continuous paths $x_n(t)$, $1 \leqslant n \leqslant N$. Then $\langle \exp \lambda \int S(\mathbf{t})^2\, d\mathbf{t} \rangle < +\infty$ if and only if $\lambda < (2\mu_1^{x(\mathbf{t})})^{-1}$, and in this case one has

$$\log \langle e^{\lambda \int S(\mathbf{t})^2\, d\mathbf{t}} \rangle = -\frac{1}{2} \sum_{i \geqslant 1} \log (1 - 2\lambda \mu_i^{x(\mathbf{t})}) \leqslant \frac{N\lambda \int_0^T C(0, t, t)\, dt}{1 - 2\lambda \mu_1^{x(\mathbf{t})}}. \tag{7}$$

To show (7), consider the Hilbert space of the $L^2(d\mathbf{t})$ functions $\varphi(n,t) \equiv \varphi(\mathbf{t})$ with the scalar product (φ, ψ). Since $C(\mathbf{s},\mathbf{t})$ is continuous in $(\mathbf{s},\mathbf{t})$, and therefore bounded in compact sets, we have that $\int\int C(\mathbf{s},\mathbf{t})^2\, d\mathbf{s}d\mathbf{t} < +\infty$. By ref. 5, Theorem VI.23, it follows that the covariance operator is compact (and self-adjoint) in $L^2(d\mathbf{t})$. Therefore there is an orthonormal basis $\{\varphi_j\}_{j\geqslant 1}$ such that $\hat{T}_C\varphi_j = \mu_j^{x(\mathbf{t})}\varphi_j$. Consider now the sequence of random variables $X_j = (S, \varphi_j)$. As linear functionals of the Gaussian field S, the X_j's form a Gaussian sequence with $\langle X_i\rangle = 0$ and $\langle X_i X_j\rangle = (\varphi_i, \hat{T}_C\varphi_j) = \mu_j^{x(\mathbf{t})}\delta_{ij}$. The equality in Eq. (7) is then obtained straightforwardly from $\int S^2(\mathbf{t})\, d\mathbf{t} = \sum_{j=1}^{+\infty} X_j^2$ and the simple Gaussian identity $\langle e^{\lambda X_i^2}\rangle = (1-2\lambda\mu_i^{x(\mathbf{t})})^{-1/2}$, for $2\lambda\mu_i^{x(\mathbf{t})} < 1$. The inequality in Eq. (7) follows from $-\log(1-x) \leqslant x/(1-x)$ and the fact that $\sum_i \mu_i^{x(\mathbf{t})} = \int C(\mathbf{t},\mathbf{t})\, d\mathbf{t} = N\int_0^T C(0,t,t)\, dt$.

In the following subsection, $\varphi(\mathbf{t}) \equiv \varphi(n,t)$ will denote a set of N test functions normalized such that $(\varphi,\varphi) = \sum_{n=1}^N \int_0^T \varphi(n,t)^2\, dt = 1$.

Lemma 2. Assume that for every $T > 0$ one has $\lim_{x\to 0} \sup_{s,t\in[0,T]} |C(x,s,t) - C(0,s,t)| = 0$. Then, $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$|(\varphi, \hat{T}_C\varphi) - (\varphi, \hat{T}_{C_0}\varphi)| < \varepsilon$$

for every $x_n(\cdot) \in B_{\delta,T}$, $1 \leqslant n \leqslant N$, where $B_{\delta,T}$ is the set of continuous paths $x(\cdot)$ such that $|x(t)| < \delta$ for every $t \in [0,T]$.

The proof of this lemma is straightforward: from the uniform convergence condition on $C(x,s,t)$ it follows that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $|C(\mathbf{s},\mathbf{t}) - C_0(\mathbf{s},\mathbf{t})| < \varepsilon$ for every $x_n(\cdot) \in B_{\delta,T}$, $1 \leqslant n \leqslant N$. Thus, $\forall\varepsilon' > 0$, $\exists\delta > 0$ such that

$$\begin{aligned} |(\varphi, \hat{T}_C\varphi) - (\varphi, \hat{T}_{C_0}\varphi)| &\leqslant (|\varphi|, \hat{T}_{|C-C_0|}\, |\varphi|) \\ &< \varepsilon'\left(\int |\varphi(\mathbf{s})|\, d\mathbf{s}\right)^2 \leqslant \varepsilon' NT, \end{aligned}$$

for every $x_n(\cdot) \in B_{\delta,T}$, $1 \leqslant n \leqslant N$. It remains to take $\varepsilon' = \varepsilon/(NT)$, which proves Lemma 2.

We can now state the main result of this section. Namely, that one of the diffusion effects on the divergence of the moments of $\mathscr{E}(x,T)$ is a lowering (or, more exactly, a non-increasing) of the critical coupling. The rigorous formulation of this result can be stated as the following proposition.

Proposition 1. For every $T>0$, if $\lim_{x\to 0}\sup_{s,t\in[0,T]}|C(x,s,t)-C(0,s,t)|=0$, then $\lambda_{cN}(T)\leqslant\bar{\lambda}_{cN}(T)$.

In order to prove this proposition, one writes the moments of $\mathscr{E}$ in terms of the Feynman–Kac formula

$$\langle\mathscr{E}(0,T)^N\rangle=\left\langle\left\langle\exp\left[\lambda\int S(\mathbf{t})^2\,d\mathbf{t}\right]\right\rangle\right\rangle_{x(\mathbf{t})},\tag{8}$$

where $\langle\cdot\rangle_{x(\mathbf{t})}$ denotes a N-fold Wiener integral over N Brownian paths $x_n(t)$, $1\leqslant n\leqslant N$, each arriving at $x=0$. Let $\lambda>\bar{\lambda}_{cN}(T)$, i.e. $\mu_1>(2N\lambda)^{-1}$, where μ_1 is the largest eigenvalue of the covariance operator $\hat{\tau}_{C_0}$. Let $\phi_1(t)$ be the normalized eigenfunction associated with μ_1, and $\phi(\mathbf{t})\equiv\phi(n,t)=N^{-1/2}\phi_1(t)$ for every $1\leqslant n\leqslant N$. [N.B.: the factor $N^{-1/2}$ ensures the normalization $(\phi,\phi)=1$]. By definition of $\mu_1^{x(\mathbf{t})}$, one has

$$\mu_1^{x(\mathbf{t})}\geqslant(\phi,\hat{T}_C\phi).\tag{9}$$

By Lemma 2, $\forall\varepsilon>0$, $\exists\delta>0$ such that

$$(\phi,\hat{T}_C\phi)\geqslant(\phi,\hat{T}_{C_0}\phi)-\varepsilon=N\mu_1-\varepsilon\tag{10}$$

for every $x_n(\cdot)\in B_{\delta,T}$, $1\leqslant n\leqslant N$. If one now takes $\varepsilon<N\mu_1-\frac{1}{2\lambda}$, it follows from Eqs. (9) and (10) that $\mu_1^{x(\mathbf{t})}>1/2\lambda$ and so, by Lemma 1,

$$\left\langle\exp\left[\lambda\int S(\mathbf{t})^2\,d\mathbf{t}\right]\right\rangle=+\infty$$

for every $x_n(\cdot)\in B_{\delta,T}$, $1\leqslant n\leqslant N$. Finally, since the set of the Brownian paths $x_n(t)$ that are in $B_{\delta,T}$ has a strictly positive Wiener measure, one finds from Eq. (8) that $\langle\mathscr{E}(0,T)^N\rangle=+\infty$, so $\lambda\geqslant\lambda_{cN}(T)$ which proves the proposition 1.

Note that imposing the uniform convergence of $C(x,s,t)$ to $C(0,s,t)$ is not a very restrictive condition. As far as we know, it seems to be fulfilled by any nonpathological stochastic field S of physical interest.

IV. EQUALITY OF $\lambda_{cN}(T)$ AND $\bar{\lambda}_{cN}(T)$ FOR A CLASS OF S

For a large class of Gaussian fields S it is possible to prove that diffusion has no effect on the onset of the divergence of $\langle\mathscr{E}(x,T)^N\rangle$, i.e. $\lambda_{cN}(T)=\bar{\lambda}_{cN}(T)$.

Proposition 2. Assume that for every $T>0$ one has $\lim_{x\to 0}\sup_{s,t\in[0,T]}|C(x,s,t)-C(0,s,t)|=0$, and that $|C(x,s,t)|\leqslant C(0,s,t)$ for every $x\in\mathbb{R}^d$ and $s,t\in[0,T]$. Then $\lambda_{cN}(T)=\bar{\lambda}_{cN}(T)$.

The proof of this proposition is as follows: By the uniform convergence condition on $C(x, s, t)$ and Proposition 1 one already has $\lambda_{cN}(T) \leqslant \bar{\lambda}_{cN}(T)$. It remains to show that $\bar{\lambda}_{cN}(T) \leqslant \lambda_{cN}(T)$. Let μ_1 be the largest eigenvalue of the covariance operator $\hat{t}_{C_0}$. Let $\phi_1(\mathbf{t})$ be a principal (normalized) eigenvector for the covariance operator $\hat{T}_C$. One has

$$\mu_1^{x(\mathbf{t})} = (\phi_1, \hat{T}_C \phi_1) \leqslant (|\phi_1|, |\hat{T}_C \phi_1|) \leqslant (|\phi_1|, \hat{T}_{C_0} |\phi_1|) \leqslant N \mu_1,$$

where the second inequality follows from the condition $|C(x, s, t)| \leqslant C(0, s, t)$. Suppose now $\lambda < \bar{\lambda}_{cN}(T)$, i.e. $\lambda < (2N\mu_1)^{-1}$. Then $\lambda < (2\mu_1^{x(\mathbf{t})})^{-1}$ and by Lemma 1

$$\left\langle \exp\left[\lambda \int S(\mathbf{t})^2\, dt\right]\right\rangle \leqslant \exp\left[\frac{N\lambda \int_0^T C(0, t, t)\, dt}{1 - 2\lambda \mu_1^{x(\mathbf{t})}}\right] \leqslant \exp\left(\frac{N\lambda \int_0^T C(0, t, t)\, dt}{1 - 2N\lambda \mu_1}\right).$$

Since this inequality is uniform over all Brownian paths, we finally have

$$\langle \mathscr{E}(0, T)^N \rangle \leqslant \exp\left(\frac{N\lambda \int_0^T C(0, t, t)\, dt}{1 - 2N\lambda \mu_1}\right) < +\infty,$$

and therefore $\lambda < \lambda_{cN}(T)$, which proves the proposition 2.

This result shows that for Gaussian fields S fulfilling the not so restrictive conditions of Proposition 2, it is sufficient to solve the diffusion-free problem to determine the onset of the divergence of $\langle \mathscr{E}(x, T)^N \rangle$. It is therefore interesting to show how such fields can be actually obtained. To this end, the remaining of this section will be devoted to explicitely construct two typical examples of stochastic fields S which fulfill the conditions of Proposition 2.

A. An Example of Nonstationary *S*

The first example is the diffusive counterpart of the Gaussian field defined by Eq. (2). Let $S(x, t)$ be the solution to

$$\begin{cases} \partial_t S(x, t) - \frac{1}{2} \Delta S(x, t) = 0, \\ t \in [0, T],\ x \in \mathbb{R}^d, \text{ and } S(x, 0) = \mathscr{S}(x), \end{cases} \tag{11}$$

where $\mathscr{S}(x)$ is a real homogeneous Gaussian field defined by

$$\begin{aligned} \langle \mathscr{S}(x) \rangle &= 0, \\ \langle \mathscr{S}(x)\, \mathscr{S}(x') \rangle &= \mathscr{C}(x - x'), \end{aligned} \tag{12}$$

with $\mathscr{C}(x)$ a given[3] function of x normalized such that $\mathscr{C}(0) \equiv \langle S(x, 0)^2 \rangle = 1$. One has

$$S(x, t) = \int \mathscr{S}(k)\, \mathrm{e}^{ikx - \frac{1}{2}k^2 t} d^d k, \tag{13}$$

where $\mathscr{S}(k)$ is the Fourier transform of $\mathscr{S}(x)$, and from Eqs. (12) and (13) it follows that $S(x, t)$ is a real homogeneous nonstationary Gaussian field with

$$\begin{aligned} \langle S(x, t) \rangle &= 0, \\ \langle S(x, t)\, S(x', t') \rangle &= \int \mathscr{C}(k)\, \mathrm{e}^{ik(x-x') - \frac{1}{2}k^2(t+t')} d^d k, \end{aligned} \tag{14}$$

where $\mathscr{C}(k)$ is the Fourier transform of $\mathscr{C}(x)$. Since $\mathscr{C}(k)$ is real and positive,[3] one has

$$\begin{aligned} |C(x, s, t)| &\equiv |\langle S(x, s) S(0, t) \rangle| \\ &= \left| \int \mathscr{C}(k)\, \mathrm{e}^{ikx - \frac{1}{2}k^2(s+t)} d^d k \right| \\ &\leqslant \int \mathscr{C}(k)\, \mathrm{e}^{-\frac{1}{2}k^2(s+t)} d^d k = C(0, s, t), \end{aligned}$$

for every $x \in \mathbb{R}^d$ and $s, t \in [0, T]$, so $S(x, t)$ fulfills the conditions of Proposition 2.

B. An Example of Stationary *S*

The second example is provided by a modified version of Eq. (11) obtained by adding a source term à la Langevin on its right-hand side. Namely, let $S(x, t)$ be the solution to

$$\begin{cases} \partial_t S(x, t) - \frac{1}{2} \Delta S(x, t) = L(x, t), \\ t \in \left]-\infty, T\right], x \in \mathbb{R}^d, \text{ and } S(x, -\infty) = 0, \end{cases} \tag{15}$$

where the Langevin source term $L(x, t)$ is a homogeneous Gaussian white noise defined by

$$\begin{aligned} \langle L(x, t) \rangle &= 0, \\ \langle L(x, t)\, L(x', t') \rangle &= -\delta(t-t')\, \Delta_x \mathscr{C}(x-x'), \end{aligned} \tag{16}$$

with $\mathscr{C}(x)$ a given[3] function of x normalized such that $\mathscr{C}(0) = 1$. The solution to Eq. (15) reads

$$S(x, t) = \int d^d k \left[\mathrm{e}^{ikx} \int_{-\infty}^{t} \mathrm{e}^{-\frac{1}{2}k^2(t-s)} L(k, s)\, ds \right], \tag{17}$$

where $L(k, t)$ is the Fourier transform of $L(x, t)$. From Eqs. (16) and (17) it can be shown that $S(x, t)$ is a real homogeneous stationary Gaussian field with

$$\begin{aligned} &\langle S(x, t) \rangle = 0, \\ &\langle S(x, t)\, S(x', t') \rangle = \int \mathscr{C}(k)\, \mathrm{e}^{ik(x-x') - \frac{1}{2}k^2 |t-t'|} d^d k, \end{aligned} \tag{18}$$

where $\mathscr{C}(k)$ is the Fourier transform of $\mathscr{C}(x)$. As previously, since $\mathscr{C}(k)$ is real and positive,[3] one has

$$\begin{aligned} |C(x, s, t)| &\equiv |\langle S(x, s)\, S(0, t) \rangle| \\ &= \left| \int \mathscr{C}(k)\, \mathrm{e}^{ikx - \frac{1}{2}k^2 |s-t|} d^d k \right| \\ &\leqslant \int \mathscr{C}(k)\, \mathrm{e}^{-\frac{1}{2}k^2 |s-t|} d^d k = C(0, s, t), \end{aligned}$$

for every $x \in \mathbb{R}^d$ and $s, t \in [0, T]$, and so $S(x, t)$ fulfills the conditions of Proposition 2.

More generally, it can be checked that any real homogeneous Gaussian field $S(x, t)$ defined by

$$\begin{aligned} &\langle S(x, t) \rangle = 0, \\ &\langle S(x, t)\, S(x', t') \rangle = \int \mathscr{C}(k, t, t')\, \mathrm{e}^{ik(x-x')} d^d k, \end{aligned}$$

where $\mathscr{C}(k, t, t')$ is real and positive, fulfills the conditions of Proposition 2.

V. EXPLICIT SOLUTION OF THE DIFFUSION-FREE PROBLEM FOR A CLASS OF *S*

In this section we show that an explicit computation of the diffusion-free amplification factor $\langle \exp(N\lambda \int_0^T S(0, t)^2\, dt) \rangle$ can be achieved if $S(0, t)$ is a linear functional of a Gauss–Markov process. Note that determining

$\bar{\lambda}_{cN}(T)$ amounts to finding the largest eigenvalue of the covariance operator $\hat{t}_{C_0}$, which in principle can always be achieved, at least numerically. As shown above, $\bar{\lambda}_{cN}(T) \geqslant \lambda_{cN}(T)$ with equality holding when Proposition 2 is applicable. Since $\bar{\lambda}_{cN}(T) = N^{-1}\bar{\lambda}_{c1}(T)$ in the diffusion free case, we will take $N = 1$ in the remaining of this section without loss of generality.

A. Solution of the Diffusion-Free Problem using the Feynman–Kac Formula

We consider the case where the Gaussian process $S(0, t)$ can be written as

$$S(0, t) = \langle c, Y(t) \rangle, \tag{19}$$

where $\langle x, y \rangle \equiv x^\dagger y = \sum_i x_i y_i$, c is a given n-dimensional vector, and $Y(t)$ is a n-dimensional Gauss–Markov process defined as the solution to the linear stochastic differential equation

$$\begin{cases} dY(t) + AY(t)\, dt = G dB(t), \\ Y(0) \text{ Gaussian with } \langle Y(0) \rangle = 0. \end{cases} \tag{20}$$

Here, A and G are constant $n \times n$ matrices, and $B(t)$ is a n-dimensional Brownian motion. From Eqs. (19) and (20), it follows that one can write the diffusion-free amplification factor as a Feynman–Kac formula

$$\langle e^{\lambda \int_0^T S(0,t)^2\, dt} \rangle = \langle e^{\lambda \int_0^T \langle Y(t), CY(t) \rangle\, dt} \rangle = \int v(y, T)\, d^n y, \tag{21}$$

where C denotes the symmetrical $n \times n$ matrix $c \otimes c$, and $v(y, t)$ is the solution to the parabolic equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = (\mathrm{Tr} A + \lambda \langle y, Cy \rangle)\, v + \langle Ay, \nabla \rangle v + \dfrac{1}{2} \langle G^\dagger \nabla, G^\dagger \nabla \rangle\, v, \\ v(y, 0) = \left(\dfrac{1}{2\pi} \right)^{n/2} \dfrac{1}{\sqrt{|U|}} \exp\left[-\dfrac{1}{2} \langle y, U^{-1} y \rangle \right], \end{cases} \tag{22}$$

with $U = \mathrm{Cov}[Y(0), Y(0)]$. The solution to Eq. (22) has the form

$$v(y, t) = \left(\frac{1}{2\pi} \right)^{n/2} \frac{1}{\sqrt{|K(t)|}} \exp\left[-\frac{1}{2} \langle y, K(t)^{-1}\, y \rangle + \lambda \int_0^t \mathrm{Tr} CK(s)\, ds \right], \tag{23}$$

where $K(t)$ is a symmetrical $n \times n$ matrix which is the solution to

$$\begin{cases} \dfrac{dK(t)}{dt} = GG^\dagger - [AK(t) + K(t)\,A^\dagger] + 2\lambda K(t)\,CK(t), \\ K(0) = U. \end{cases} \tag{24}$$

Thus, from Eqs. (21) and (23) one has

$$\left\langle e^{\lambda \int_0^T S(0,t)^2\,dt} \right\rangle = e^{\lambda \int_0^T \mathrm{Tr}\, CK(t)\,dt}. \tag{25}$$

with $K(t)$ given by the Riccati equation (24).

The solution to Eq. (24) is known to explode in finite time for large enough λ. For $n = 1$, in which case $S(0, t)$ is itself Markovian, Eq. (24) is solved straightforwardly (see Section 5 B). For $n \geqslant 2$, the solution to Eq. (24) can be obtained by the so-called Hamiltonian method: we define the $2n \times 2n$ matrix

$$H = \begin{pmatrix} A^\dagger & -2\lambda C \\ GG^\dagger & -A \end{pmatrix}$$

and solve the linear differential equation

$$\frac{d}{dt} \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} = H \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix}, \tag{26}$$

with the initial condition

$$\begin{bmatrix} Q(0) \\ P(0) \end{bmatrix} = \begin{bmatrix} I \\ U \end{bmatrix}.$$

The solution $K(t)$ to the Riccati equation (24) is easily checked to be given by

$$K(t) = P(t)\, Q(t)^{-1}, \tag{27}$$

which explodes if and only if $Q(t)$ becomes singular.[6] Since Eq. (26) is a linear equation, it can in principle be solved by a symbolic computation program.

B. Application to the $n=1$ Case

As an example, let us consider the simplest case $n=1$ with $C(0, t, t') = e^{-|t-t'|}$. In this limit, the diffusion-free amplification factor reads

$$\langle e^{\lambda \int_0^T S(0,t)^2\, dt} \rangle = \langle e^{\lambda \int_0^T Y(t)^2\, dt} \rangle = e^{\lambda \int_0^T K(t)\, dt}, \tag{28}$$

where $Y(t)$ is the Ornstein–Uhlenbeck process

$$\begin{cases} dY(t) + Y(t)\, dt = \sqrt{2}\, dB(t), \\ \langle Y(0) \rangle = 0,\ \langle Y(0)^2 \rangle = 1, \end{cases} \tag{29}$$

and $K(t)$ is the solution to the Riccati equation

$$\begin{cases} \dfrac{1}{2}\dfrac{dK(t)}{dt} = 1 - K(t) + \lambda K(t)^2, \\ K(0) = 1. \end{cases} \tag{30}$$

Equation (30) can be easily solved by means of the substitution $2\lambda K(t) = -d \log u(t)/dt$. Inserting the result into Eq. (28), one obtains

$$\langle e^{\lambda \int_0^T S(0,t)^2\, dt} \rangle = \frac{e^{T/2}}{\sqrt{\cosh(\alpha T) + \alpha^{-1}(1-2\lambda)\sinh(\alpha T)}}, \qquad \lambda < 1/4, \tag{31}$$

$$\langle e^{\lambda \int_0^T S(0,t)^2\, dt} \rangle = \frac{e^{T/2}}{\sqrt{1+T/2}}, \qquad \lambda = 1/4, \tag{32}$$

$$\langle e^{\lambda \int_0^T S(0,t)^2\, dt} \rangle = \frac{e^{T/2}}{\sqrt{\cos(\alpha T) + \alpha^{-1}(1-2\lambda)\sin(\alpha T)}}, \qquad \lambda > 1/4, \tag{33}$$

where $\alpha = |1-4\lambda|^{1/2}$. It can be seen from Eq. (33) that, for $\lambda > 1/4$, $\langle \exp(\lambda \int_0^T S(0,t)^2\, dt) \rangle$ diverges as T tends (from below) to the critical time $T_c(\lambda)$ given by

$$T_c(\lambda) = \frac{1}{\sqrt{4\lambda - 1}} \tan^{-1}\left(\frac{\sqrt{4\lambda-1}}{2\lambda - 1} \right), \tag{34}$$

where the determination of $\tan^{-1}$ is such that $0 < \tan^{-1} \leqslant \pi$. Inverting Eq. (34) and using $\bar{\lambda}_{cN}(T) = N^{-1}\bar{\lambda}_{c1}(T)$ gives the diffusion-free critical coupling $\bar{\lambda}_{cN}(T)$ in the cases where $C(0, t, t') = e^{-|t-t'|}$.[7]

VI. DEPENDENCE OF THE CRITICAL COUPLING ON SPACE DIMENSIONALITY

In this section we study the dependence of $\lambda_{cN}(T)$ on the space dimensionality D. We will restrict ourselves to the cases where the correlation function C can be written out as

$$C_D(x, t, t') = C_d(x_\parallel, t, t')\, C_{D-d}(x_\perp, t, t'), \tag{35}$$

with C_D, C_d and C_{D-d} continuous, symmetric, and positive definite, and where $x_\parallel$ is the projection of x onto a d-dimensional subspace ($d < D$) and $x_\perp = x - x_\parallel$. In the following, a correlation function of this type will be called a factorable correlation function. It is worth noting that such a correlation function can be very easily obtained, e.g. when the Gaussian field S is defined by either Eq. (14) or Eq. (18) in the cases where $\mathscr{C}(k)$ is factorable as $\mathscr{C}(k) = \mathscr{C}_d(k_\parallel)\, \mathscr{C}_{D-d}(k_\perp)$.

We prove that as λ increases, the divergence of $\langle \mathscr{E}(x, T)^N \rangle$ obtained in the original D-dimensional problem cannot occur before the one obtained in the projected d-dimensional problem whenever $0 \leqslant C_{D-d}(0, t, t) \leqslant 1$. Since many stochastic fields S of physical interest, e.g. in optics, do have a factorable correlation function, we expect this result to be useful for the comparison of two-dimensional numerical simulations with experiments and three-dimensional numerical simulations. Before expressing this result in a more rigorous way, we begin with two technical lemmas that will be needed in the following.

Lemma 3. Consider a D-dimensional problem and let $\mu_1^{x(\mathbf{t})}$ be the largest eigenvalue of the covariance operator $\hat{T}_{C_D}$ and N given continuous paths $x(\mathbf{t})$. Then $\lambda_{cN}(T, D) = [2 \sup_{x(\mathbf{t})} \mu_1^{x(\mathbf{t})}]^{-1}$.

This lemma can be proven straightforwardly by successively considering the inequalities $\lambda > [2 \sup_{x(\mathbf{t})} \mu_1^{x(\mathbf{t})}]^{-1}$ and $\lambda < [2 \sup_{x(\mathbf{t})} \mu_1^{x(\mathbf{t})}]^{-1}$, and by following the same lines of reasoning as for the proofs of Propositions 1 and 2 respectively, where one replaces the N paths $x(\mathbf{t}) = 0$ corresponding to $\bar{\lambda}_{cN}(T) = [2\mu_1^{x(\mathbf{t})=0}]^{-1}$ by N paths maximizing $\mu_1^{x(\mathbf{t})}$.[8]

Lemma 4. Let $K_0(\mathbf{s}, \mathbf{t})$, $K_1(\mathbf{s}, \mathbf{t})$, and $K_2(\mathbf{s}, \mathbf{t})$ be three symmetric kernels such that: (i) $K_0(\mathbf{s}, \mathbf{t}) = K_1(\mathbf{s}, \mathbf{t})\, K_2(\mathbf{s}, \mathbf{t})$; (ii) K_2 is a positive definite continuous symmetric kernel; (iii) $0 \leqslant K_2(\mathbf{t}, \mathbf{t}) < 1$ and the largest eigenvalue of K_1 is positive, or $K_2(\mathbf{t}, \mathbf{t}) = 1$ and no condition on the sign of the largest eigenvalue of K_1. Then $\mu_1(K_0) \leqslant \mu_1(K_1)$, where $\mu_1(K_\alpha)$ denotes the largest eigenvalue of K_α.

The proof of this lemma is as follows: since K_2 is a positive definite continuous symmetric kernel, Mercer's theorem holds[9] and this kernel admits the expansion

$$K_2(\mathbf{s}, \mathbf{t}) = \sum_i a_i f_i(\mathbf{s})\, f_i(\mathbf{t}), \tag{36}$$

where $a_i \geqslant 0$ and $f_i(\mathbf{t})$ respectively denote the i^{th} eigenvalue of the operator $\hat{T}_{K_2}$ and the associated normalized eigenfunction. Let $\phi_1(\mathbf{t})$ be a principal (normalized) eigenfunction of the operator $\hat{T}_{K_0}$ and $\mu_1(K_0)$ the corresponding largest eigenvalue. From the condition (i) and Eq. (36), one has

$$\mu_1(K_0) = (\phi_1, \hat{T}_{K_0}\phi_1) = \sum_i a_i (f_i\phi_1, \hat{T}_{K_1} f_i\phi_1) = \sum_i a_i M_i (\eta_i, \hat{T}_{K_1}\eta_i), \tag{37}$$

where M_i and $\eta_i(\mathbf{t})$ are given by

$$M_i = (f_i\phi_1, f_i\phi_1),$$

and

$$\eta_i(\mathbf{t}) = M_i^{-1/2} f_i(\mathbf{t})\, \phi_1(\mathbf{t}),$$

such that $(\eta_i, \eta_i) = 1$. By the definition of $\mu_1(K_1)$ and from $K_2(\mathbf{t}, \mathbf{t}) \leqslant 1$, condition (iii), one has respectively

$$\mu_1(K_1) \geqslant (\eta_i, \hat{T}_{K_1}\eta_i), \tag{38}$$

and

$$\begin{aligned} \sum_i a_i M_i &= \int \left[\sum_i a_i f_i(\mathbf{t})^2 \right] \phi_1(\mathbf{t})^2 \, d\mathbf{t} \\ &= \int K_2(\mathbf{t}, \mathbf{t})\, \phi_1(\mathbf{t})^2 \, d\mathbf{t} \leqslant \int \phi_1(\mathbf{t})^2 \, d\mathbf{t} = 1. \end{aligned} \tag{39}$$

So, from Eqs. (37), (38), (39) and the condition (iii), it follows that $\mu_1(K_0) \leqslant \mu_1(K_1)$, which proves Lemma 4.

We can now proceed to rigorously express and prove the result stated at the beginning of this section. Let $\lambda_{cN}(T, D)$ be the critical coupling associated with a D-dimensional problem in which the correlation function of the Gaussian field S is given by C_D. One has the following proposition:

Proposition 3. For every $T > 0$, if $C_D(x, t, t')$ is a factorable correlation function such that $0 \leqslant C_{D-d}(0, t, t) \leqslant 1$ for $0 \leqslant t \leqslant T$, then $\lambda_{cN}(T, D) \geqslant \lambda_{cN}(T, d)$.

The proof of this proposition is straightforward. By the definition of a factorable correlation function one has $C_D(\mathbf{s}, \mathbf{t}) = C_d(\mathbf{s}, \mathbf{t})\, C_{D-d}(\mathbf{s}, \mathbf{t})$, where both $C_d(\mathbf{s}, \mathbf{t})$ and $C_{D-d}(\mathbf{s}, \mathbf{t})$ are continuous, symmetric, and positive definite. Since $C_{D-d}(\mathbf{t}, \mathbf{t}) \equiv C_{D-d}(0, t, t)$ and $0 \leqslant C_{D-d}(0, t, t) \leqslant 1$ by assumption, one can apply the lemma 4 with $K_0 = C_D$, $K_1 = C_d$, and $K_2 = C_{D-d}$. It follows immediately that $\mu_1^{x(\mathbf{t})} \leqslant \tilde{\mu}_1^{x_\parallel(\mathbf{t})}$, where $\tilde{\mu}_1^{x_\parallel(\mathbf{t})}$ denotes the largest eigenvalue of the operator $\hat{T}_{C_d}$. Let $x_{\max}(\mathbf{t})$ be N paths maximizing $\mu_1^{x(\mathbf{t})}$.[8] One has $\sup_{x(\mathbf{t})} \mu_1^{x(\mathbf{t})} = \mu_1^{x_{\max}(\mathbf{t})} \leqslant \tilde{\mu}_1^{x_{\max\parallel}(\mathbf{t})}$, from which it follows that $\sup_{x(\mathbf{t})} \mu_1^{x(\mathbf{t})} \leqslant \sup_{x_\parallel(\mathbf{t})} \tilde{\mu}_1^{x_\parallel(\mathbf{t})}$ and, by Lemma 3, $\lambda_{cN}(T, D) \geqslant \lambda_{cN}(T, d)$, which proves the Proposition 3.

VII. SUMMARY AND PERSPECTIVES

In this paper, we have studied the effects of diffusion on the divergence of the moments of the solution to a linear amplifier driven by the square of a Gaussian field. We first proved that the divergence yielded by a diffusion-free calculation cannot occur at a smaller coupling constant than the one obtained from the full calculation (i.e. with diffusion). Then we have shown that, in the case where the absolute value of the (uniformly continuous) pump field correlation function is bounded from above by its one-point value, there is no diffusion effect on the onset of the divergence which is therefore given by a diffusion-free calculation. In this context, we have solved the diffusion-free problem explicitly when the pump field is a linear functional of a Gauss–Markov process. Finally, we have studied the dependence of the critical coupling on the space dimensionality in the case of a factorable correlation function. In particular, we have proved that the divergence obtained in a D-dimensional problem cannot occur at a smaller coupling constant than the one obtained in the projected d-dimensional problem ($d < D$).

As mentioned in the introduction, we would like to extend our results for the diffusion-amplification model (5) to the more difficult diffraction-amplification problem (1). According to Eq. (5), the results obtained in this paper also apply, beside some minor technical modifications, if the pump field is a *complex* Gaussian field as in Eq. (1). The remaining difficulty in extending our results to Eq. (1) lies in controlling the complex Feynman path-integral, compared to that of the Feynman–Kac formula for the diffusive case. Expressing $\mathscr{E}(x, t)$ as a Feynman path-integral and averaging over the realizations of S, one cannot *a priori* exclude the possibility that destructive interference effects between different path contributions make the sum of the divergent contributions finite. Thus one cannot deduce the divergence of the moments of $\mathscr{E}(x, L)$ from that of the amplification along paths arriving at the point (x, L). However, in view of the numerical results of ref. 2, which strongly suggest a divergence in the diffractive case too, it is not unreasonable to expect that Propositions 1, 2, and 3 also apply to the diffraction-amplification problem (1).[10] Proving this conjecture is another matter and is the subject of a future work. Note that in the case of Proposition 2, the on-axis correlation function of the pump field must be real and positive, which is quite restrictive if the pump field is complex. From a practical point of view (e.g. in optics), it would therefore be very interesting to find out whether there exists an enlarged version of this proposition applying to complex on-axis correlation functions as well.

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